Problem 3.44

The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathsf{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$

where a, b, and c are real numbers.

(a) If the system starts out in the state

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

what is $|\mathcal{S}(t)\rangle$?

(b) If the system starts out in the state

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

what is $|\mathcal{S}(t)\rangle$?

Solution

The procedure for finding a finite-dimensional state vector $|\mathcal{S}(t)\rangle$ is similar to the one in Chapter 2 for finding an infinite-dimensional wave function $\Psi(x,t)$: (1) Solve the eigenvalue problem for H (the TISE), (2) write the initial state vector in terms of the normalized eigenvectors, and (3) multiply each eigenvector by the corresponding wiggle factor to get the state vector at any time t.

Solve the Eigenvalue Problem for H

Begin by writing down the TISE.

$$\hat{H}|s\rangle = E|s\rangle$$

With respect to a certain basis, the given Hamiltonian matrix H represents the Hamiltonian operator $\hat{H}$.

$$(\mathsf{H} - E\mathsf{I})|s\rangle = \mathsf{0} \tag{1}$$

Since $|s\rangle$ can't be the zero vector, the matrix in parentheses must be singular.

$$\det(\mathsf{H} - E\mathsf{I}) = 0$$

$$\begin{vmatrix} a-E & 0 & b \\ 0 & c-E & 0 \\ b & 0 & a-E \end{vmatrix} = 0$$

$$(a-E)\begin{vmatrix} c-E & 0 \\ 0 & a-E \end{vmatrix} + b\begin{vmatrix} 0 & c-E \\ b & 0 \end{vmatrix} = 0$$

Evaluate the determinants and solve for the eigenvalues, the allowed energies.

$$(a-E)(c-E)(a-E)-b(c-E)b=0$$

$$(c-E)[(a-E)^2-b^2]=0$$

$$(c-E)[(a-E)+b][(a-E)-b]=0$$

$$E=\{a-b,c,a+b\}$$

Let $E_-=a-b$ and $E_0=c$ and $E_+=a+b$. Plug each of them into equation (1) to determine the associated eigenvectors.

$$(\mathsf{H}-E_-\mathsf{I})|s_-\rangle=0 \qquad (\mathsf{H}-E_0\mathsf{I})|s_0\rangle=0 \qquad (\mathsf{H}-E_+\mathsf{I})|s_+\rangle=0$$

$$\begin{pmatrix} b & 0 & b \\ 0 & c-a+b & 0 \\ b & 0 & b \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a-c & 0 & b \\ 0 & 0 & 0 \\ b & 0 & a-c \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -b & 0 & b \\ 0 & c-a-b & 0 \\ b & 0 & -b \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} bs_1+bs_3&=0 \\ (c-a+b)s_2&=0 \end{aligned}\right\} \qquad \left.\begin{aligned} (a-c)s_1+bs_3&=0 \\ bs_1+(a-c)s_3&=0 \end{aligned}\right\} \qquad \left.\begin{aligned} -bs_1+bs_3&=0 \\ (c-a-b)s_2&=0 \end{aligned}\right\}$$

$$\left.\begin{aligned} s_3&=-s_1 \\ s_2&=0 \end{aligned}\right\} \qquad \left.\begin{aligned} s_1&=0 \\ s_3&=0 \end{aligned}\right\} \qquad \left.\begin{aligned} s_1&=s_3 \\ s_2&=0 \end{aligned}\right\}$$

Consequently,

$$|s_-\rangle=\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}=\begin{pmatrix} s_1 \\ 0 \\ -s_1 \end{pmatrix}=s_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad |s_0\rangle=\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}=\begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix}=s_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |s_+\rangle=\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}=\begin{pmatrix} s_3 \\ 0 \\ s_3 \end{pmatrix}=s_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

s_1, s_2, and s_3 are arbitrary, but for the eigenvectors to be physically relevant, they have to be normalized.

$$(s_1)^2+(0)^2+(-s_1)^2=1 \qquad (0)^2+(s_2)^2+(0)^2=1 \qquad (s_3)^2+(0)^2+(s_3)^2=1$$

$$s_1=\pm\frac{1}{\sqrt{2}} \qquad s_2=\pm1 \qquad s_3=\pm\frac{1}{\sqrt{2}}$$

Therefore, the normalized eigenvectors are

$$|s_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad |s_0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |s_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Part (a)

Now write the initial state vector as a linear combination of the normalized eigenvectors.

$$\begin{aligned} |\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= A_1|s_-\rangle + B_1|s_0\rangle + C_1|s_+\rangle \\ &= \frac{A_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + B_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{C_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{A_1}{\sqrt{2}} + \frac{C_1}{\sqrt{2}} \\ B_1 \\ -\frac{A_1}{\sqrt{2}} + \frac{C_1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

The resulting system of equations for A_1, B_1, and C_1 is

$$\begin{aligned} 0 &= \frac{A_1}{\sqrt{2}} + \frac{C_1}{\sqrt{2}} \\ 1 &= B_1 \\ 0 &= -\frac{A_1}{\sqrt{2}} + \frac{C_1}{\sqrt{2}}. \end{aligned}$$

Solving it yields $A_1 = 0$ and $B_1 = 1$ and $C_1 = 0$, which means

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |s_0\rangle.$$

Finally, multiply it by the wiggle factor $e^{-iE_0t/\hbar}$ in order to get the state vector at any later time t.

$$|\mathcal{S}(t)\rangle = |s_0\rangle e^{-iE_0t/\hbar} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-ict/\hbar}$$

Part (b)

Write the initial state vector as a linear combination of the normalized eigenvectors.

$$\begin{aligned}|\mathcal{S}(0)\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} &= A_2|s_-\rangle + B_2|s_0\rangle + C_2|s_+\rangle \\ &= \frac{A_2}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} + B_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{C_2}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \\ &= \begin{pmatrix}\frac{A_2}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} \\ B_2 \\ -\frac{A_2}{\sqrt{2}} + \frac{C_2}{\sqrt{2}}\end{pmatrix}\end{aligned}$$

The resulting system of equations for A_2, B_2, and C_2 is

$$\begin{aligned}1 &= \frac{A_2}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} \\ 0 &= B_2 \\ 0 &= -\frac{A_2}{\sqrt{2}} + \frac{C_2}{\sqrt{2}}.\end{aligned}$$

Solving it yields $A_2 = 1/\sqrt{2}$ and $B_2 = 0$ and $C_2 = 1/\sqrt{2}$, which means

$$|\mathcal{S}(0)\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}|s_-\rangle + \frac{1}{\sqrt{2}}|s_+\rangle.$$

Finally, multiply each term by the respective wiggle factor to get the state vector at any later time t.

$$\begin{aligned}|\mathcal{S}(t)\rangle &= \frac{1}{\sqrt{2}}|s_-\rangle e^{-iE_-t/\hbar} + \frac{1}{\sqrt{2}}|s_+\rangle e^{-iE_+t/\hbar} \\ &= \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} e^{-i(a-b)t/\hbar} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} e^{-i(a+b)t/\hbar} \\ &= \begin{pmatrix}\frac{e^{ibt/\hbar}+e^{-ibt/\hbar}}{2} \\ 0 \\ \frac{-e^{ibt/\hbar}+e^{-ibt/\hbar}}{2}\end{pmatrix} e^{-iat/\hbar} \\ &= \begin{pmatrix}\cos\frac{bt}{\hbar} \\ 0 \\ -i\sin\frac{bt}{\hbar}\end{pmatrix} e^{-iat/\hbar}\end{aligned}$$